Loughborough University
MATHEMATICS EDUCATION NETWORK

Developing fluency with procedures *without* using traditional exercises

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Tackling the ‘boring middles’ of mathematics lesson sequences

- There are lots of interesting, rich ways to **introduce new concepts** (e.g., NRICH, etc.)
- There are lots of interesting, **rich problems** to work on, using ideas students have previously learned

Introduce new concepts and methods → Apply these concepts and methods to solve interesting problems

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But…

Does the lesson sequence **have** to have a ‘boring middle’?

Introduce new concepts and methods → Do some practice → Apply these concepts and methods to solve interesting problems

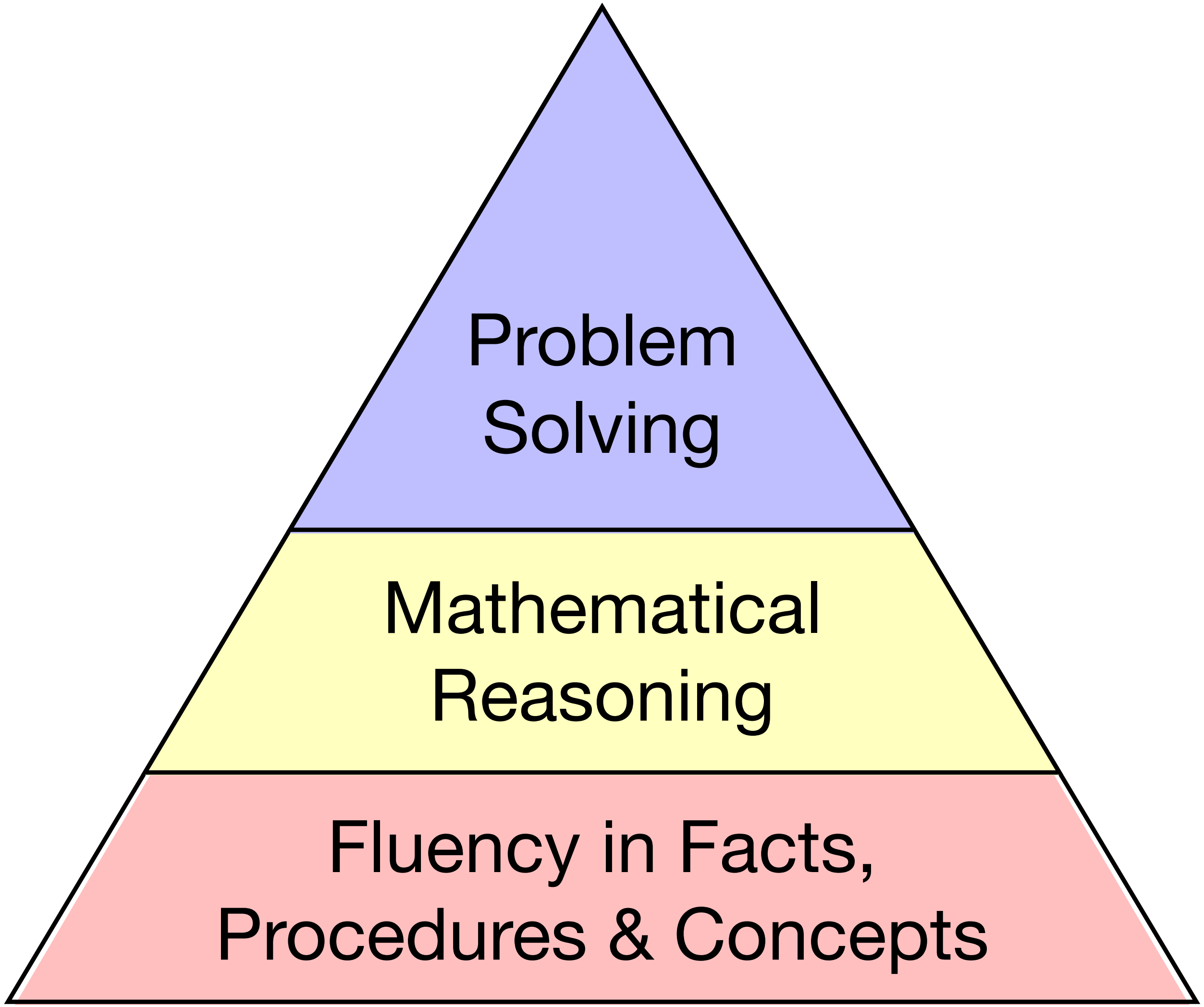
Aims for teaching mathematics

"The national curriculum for mathematics aims to ensure that all pupils:

- become ***fluent*** in the fundamentals of mathematics…
- ***reason mathematically***…
- can ***solve problems***…"

DfE (2013, p. 2, original emphasis)

There may sometimes be an overemphasis on fluency, but fluency is important, because it supports the other aims.


Problem
Solving
Mathematical
Reasoning
Fluency in Facts,
Procedures & Concepts

Does the process of developing fluency have to be *boring*?

“You have to have some boring lessons every now and again, where the students just practice something *ad nauseum*.”

What is procedural fluency?

Knowing when and how to apply a mathematical procedure and being able to perform it “accurately, efficiently, and flexibly”

(NCTM, 2014, p. 1)

This is a **good thing**.

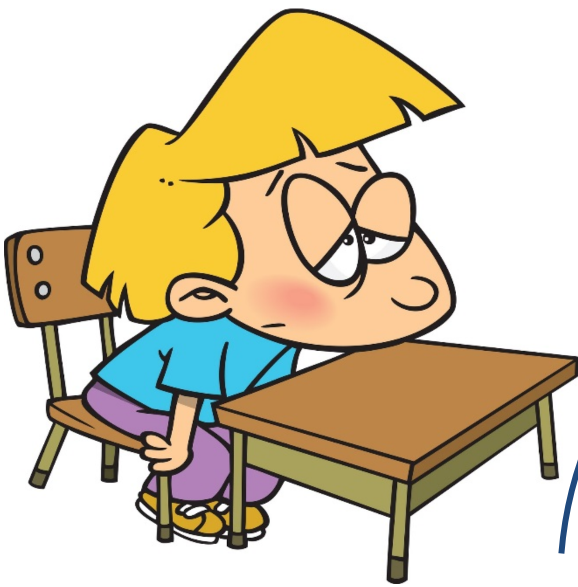
We want students to have this.

Procedural Fluency

"It is a profoundly erroneous truism … that we should cultivate the habit of thinking what we are doing. The precise opposite is the case. Mathematics ~~Civilization~~ advances by extending the number of important operations which we can perform without thinking about them."

(Whitehead, 1911, 58-61).

Standard approach to procedural fluency



Do some exercises

This may be boring

Do some more exercises

Do even more exercises

Procedural fluency

Students need **fluency** in important mathematical processes if they are to develop the expertise needed to be powerful solvers of mathematical problems.

Fluency should ***not*** be our enemy.

But, can we find more interesting ways than traditional exercises of developing students' fluency with important mathematical procedures?


Introduce new concepts and methods
Some better way of developing the necessary fluency???
Apply these concepts and methods to solve interesting problems

MATHEMATICAL FLUENCY *WITHOUT* DRILL AND PRACTICE

Colin Foster asks how can we avoid letting 'practice' dominate the teaching of the new mathematics national curriculum

Introduction

The word 'practice' appears twice in the short 'Aims' section of the *KS3 Programme of study* (DfE, 2013). The first stated aim is that all pupils:

*… become **fluent** in the fundamentals of mathematics, including through varied and frequent practice with increasingly complex problems over time, so that pupils develop conceptual understanding and the ability to recall and apply knowledge rapidly and accurately.* (p. 2)

This optimistic sentence implies that focusing on fluency will lead eventually to conceptual understanding and confidence in applying the knowledge gained. This reminds me of John Holt's (1990) observation that:

… the notion that if a child repeats a meaningless statement or process enough times it will become meaningful is as absurd as the notion that if a parrot imitates human speech long enough it will

I very much agree with. However, the following sentence, that *'Those who are not sufficiently fluent should consolidate their understanding, including through additional practice, before moving on'*, sounds to me like a recipe for never-ending, low-level, imitative rehearsing of knowledge and skills until students earn the right to anything more stimulating.

It is easy to see how students can become trapped in tedious, repetitive work, endlessly 'practising the finished product' (Prestage and Perks, 2006). Teachers are going to be told that certain students 'need more practice on X' before they are 'ready' to move on. Students will be discouraged and demotivated by constant, unimaginative repetition and the low, or slow, achievement that has led to this judgment becomes a self-fulfilling prophecy. What do we do? It is all very well for articles in *MT* to suggest rich, exciting alternatives to mechanical procedural practice, but the danger is that some of

Internalising procedures

Subordinating the skill

"practice can take place without the need for what is to be practised to become the focus of attention" Hewitt (1996, p. 34)

Give opportunities for learners to develop their fluency in important mathematical procedures while something "a bit more interesting" is going on.

'Practice through progress' (Francome & Hewitt, 2018)

MATHEMATICAL

etudes

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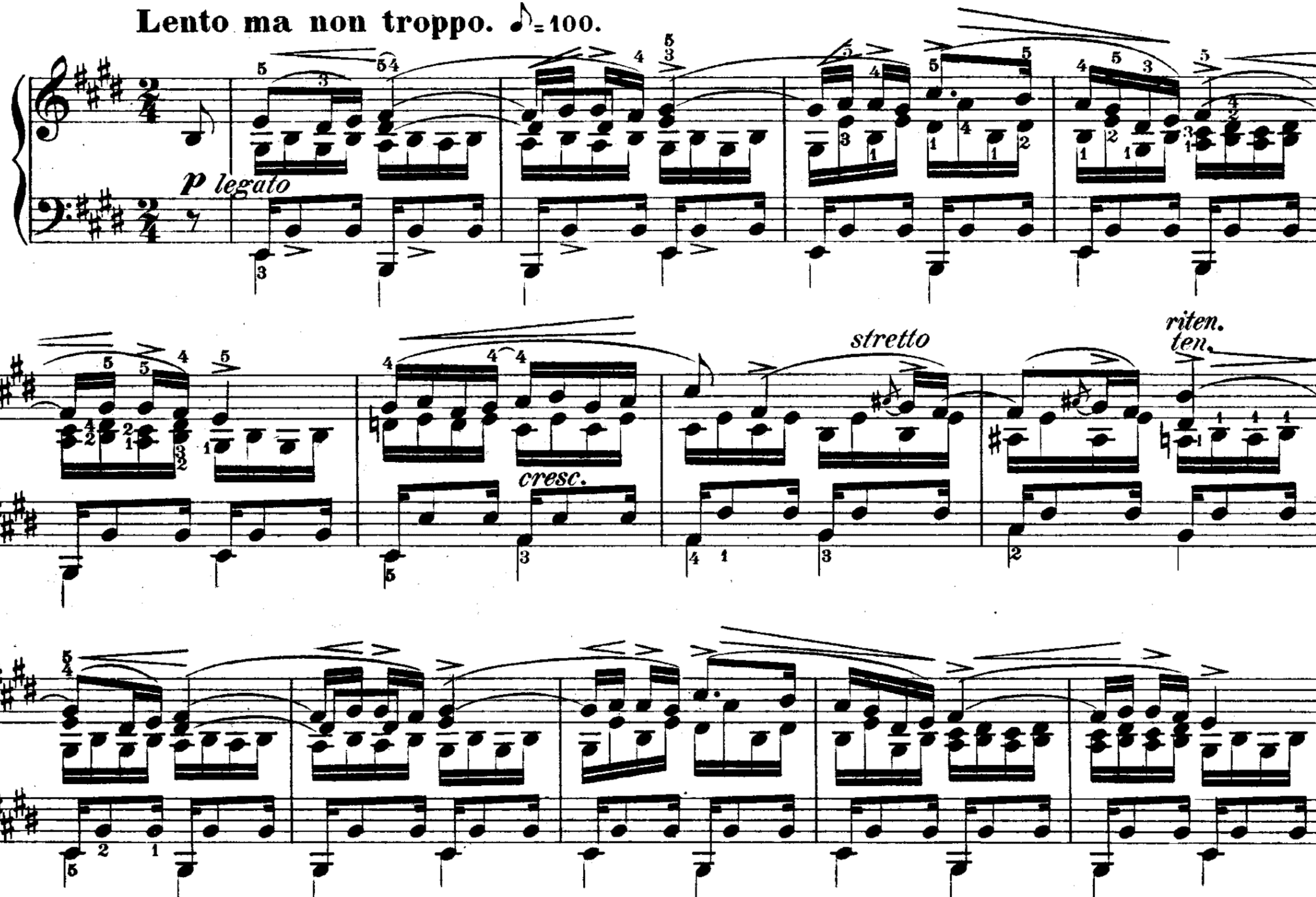
Musical Etude

“originally a study or technical exercise, later a complete and musically intelligible composition exploring a particular technical problem in an esthetically satisfying manner”

Encyclopaedia Britannica

Lento ma non troppo. ♪= 100.

3.



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>
> Phil Daro, lead author of the mathematics Common Core State Standards, used by most states in the USA

The **Mathematical Etudes Project** aims to find creative, imaginative and thought-provoking ways to help learners of mathematics develop their fluency in important mathematical procedures.

Procedural fluency involves knowing when and how to apply a procedure and being able to perform it "accurately, efficiently, and flexibly" (NCTM, 2014, p. 1). Fluency in important mathematical procedures is a critical goal within the learning of school mathematics, as security with fundamental procedures offers pupils increased power to explore more complicated mathematics at a conceptual level (Foster, 2013, 2014, 2015; Gardiner, 2014; NCTM, 2014). The new national curriculum for mathematics in England emphasises procedural fluency as the first stated aim (DfE, 2013).

But it is often assumed that the only way to get good at standard procedures is to drill and practise them *ad nauseum* using dry, uninspiring exercises.

The **Mathematical Etudes Project** aims to find practical classroom tasks which embed extensive practice of important mathematical procedures within more stimulating, rich problem-solving contexts (Foster, 2011, 2013, 2014, 2017a, 2017b). Recent research (Foster, 2017a) suggests that etudes are as good as exercises in terms of developing procedural fluency – and it seems likely that they have many other benefits in addition.

For more details see the papers listed below or scroll down for some example tasks.

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Example:

Multiplication of integers with up to 3-digits without a calculator

x 87	14 x 16	65 x 44	x 53		
17 x 17	15 x 94	56 x 64	43 x 21	9 x 99	3 x 37
77 x 48	24 x 27	95 x 40	18 x 74	66 x 96	12 x 17
97 x 34	89 x 34	73 x 10	79 x 31	86 x 62	78 x 24
79 x 67	20 x 36	23 x 45	32 x 98	34 x 24	41 x
91 x 38	78 x 52	37 x 42			

Have a go!

Making Products

Using the digits

1, 2, 3, 4 and 5,

once each, make **two** numbers which multiply to give the **biggest possible answer**.

No calculators, please!

Easy ways to start:

- Just 1, 2 and 3.

Easy extensions:

- Digits 1 to 9
- Make the smallest possible answer
- Make three numbers rather than two

Surprises:

Can you put these in order without working them out?

543×21 531×42 4321×5 542×43

Three Studies

Foster, C. (2018). Developing mathematical fluency: Comparing exercises and rich tasks. *Educational Studies in Mathematics, 97*(2), 121–141. https://doi.org/10.1007/s10649-017-9788-x

Research Question

Are etudes as effective as traditional exercises at developing students' procedural fluency or not?

Three studies:

1. Expression polygons
2. Devising equations
3. Enlargements

Quasi-experimental design

Two “parallel” classes, generally the same teacher across one lesson:

- *Control group*: complete as many short traditional exercises as possible
- *Intervention group*: tackle a mathematical etude on the same content

Pre- and post- tests administered at the beginning and end of the lesson.

Participants

	Study 1 Expression Polygons	Study 2 Devising Equations	Study 3 Enlargements	Total
N	193	194	141	528
Ages	12-14	12-14	13-15	12-15
Schools	3	5	3	11

Study	School	Location	Type	Sex	Number of students			School Total	Study Total
					Y8	Y9	Y10		
1	A	London	academy	mixed	76			76	
	B	West Midlands	academy	mixed	26	25		51	
	C	West Midlands	academy	girls	20	46		66	193
2	D	Scotland	comprehensive	mixed	29			29	
	E	London	academy	mixed	27			27	
	F	East Midlands	academy	mixed	86			86	
	G	East Midlands	academy	mixed		18		18	
	H	Kent	academy	mixed		34		34	194
3	I	West Midlands	academy	mixed		52		52	
	J	Oxfordshire	academy	mixed			47	47	
	K	West Midlands	comprehensive	mixed			42	42	141
					264	175	89	528	528

Etude 1

Expression polygons

Solving linear equations in which the unknown appears on both sides

$4x + 3 = 2x + 5$

3 $2x - 3 = x - 1$

4 $2x + 1 = 3x - 2$

5 $5x - 3 = 2x + 12$

6 $4x + 9 = 8x - 31$

7 $2x + 40 = 12x - 110$

8 $3x + 4 = 5x - 8$

9 $2x - 8 = 3x - 16$

10 $x + 1 = 5x + 9$

11 $5x = 2x + 12$

19 $3x + 9 = x - 5$

20 $6x - 4 = x + 16$

21 $x - 7 = 7x - 25$

22 $x + 5 = 4x - 4$

23 $6x + 5 = 3x - 7$

24 $x + 1 = 7x - 17$

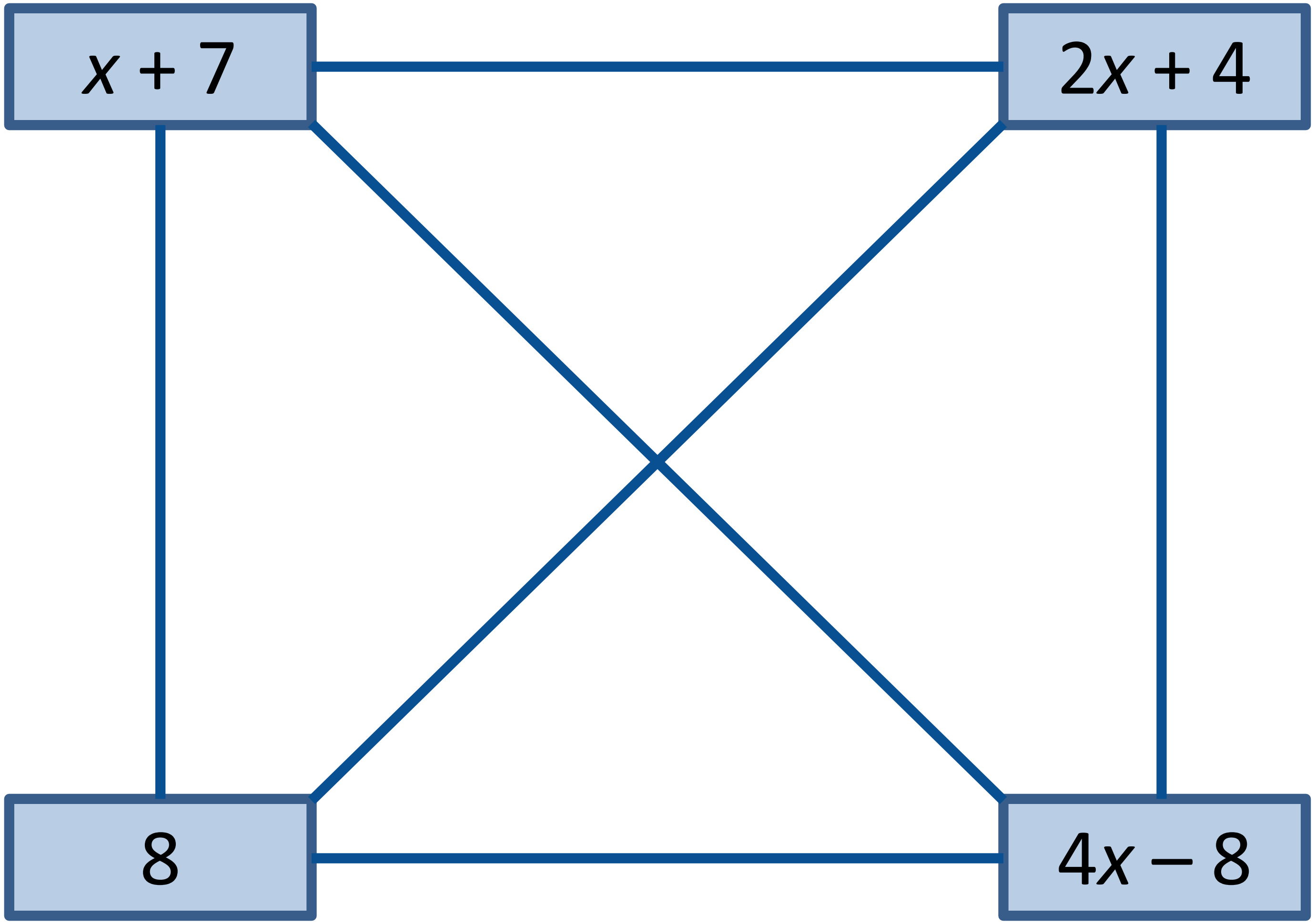
25 $3x - 4 = 5x + 6$

26 $8x + 3 = 6x + 15$

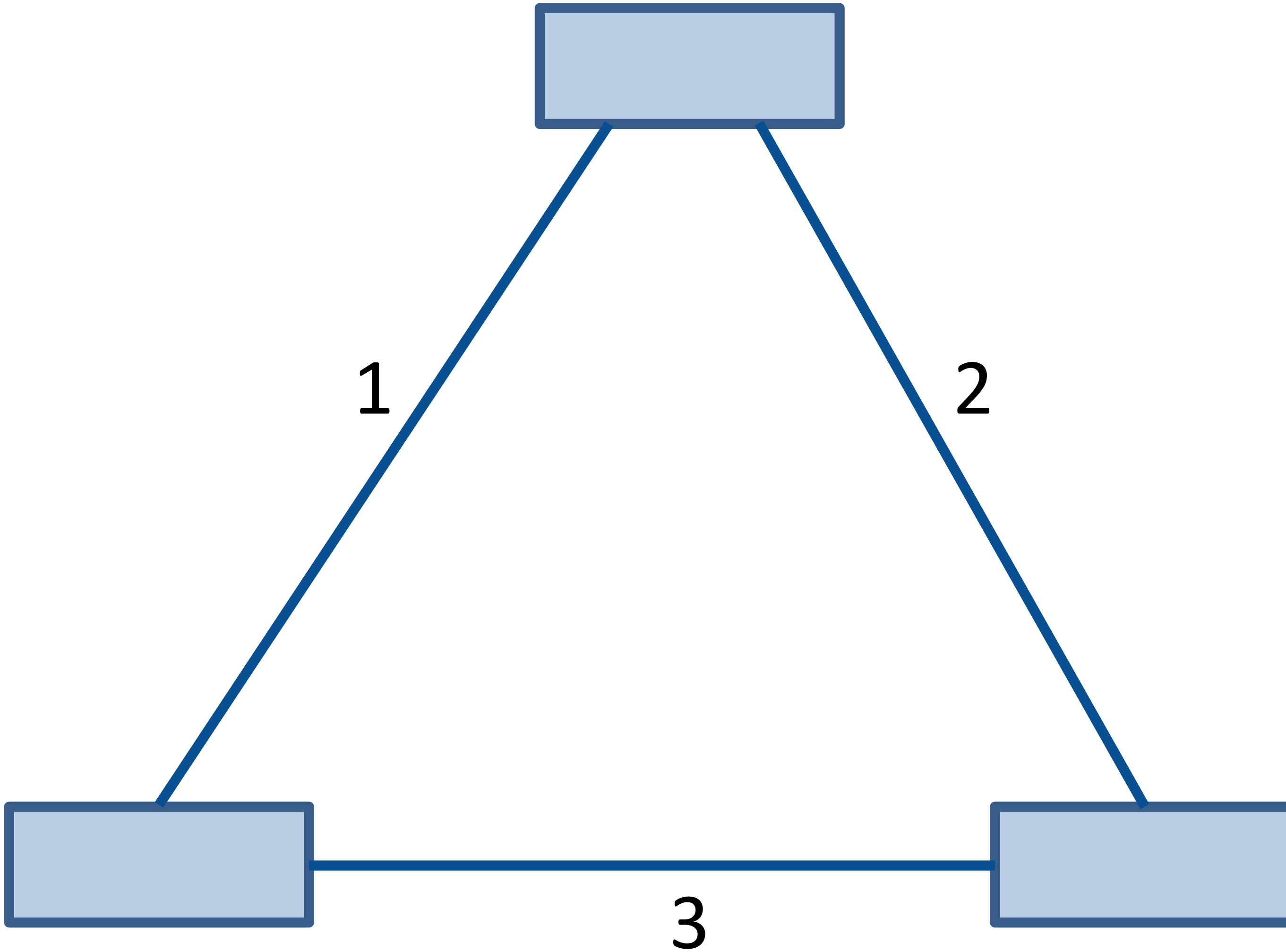
27 $x = 20 - x$

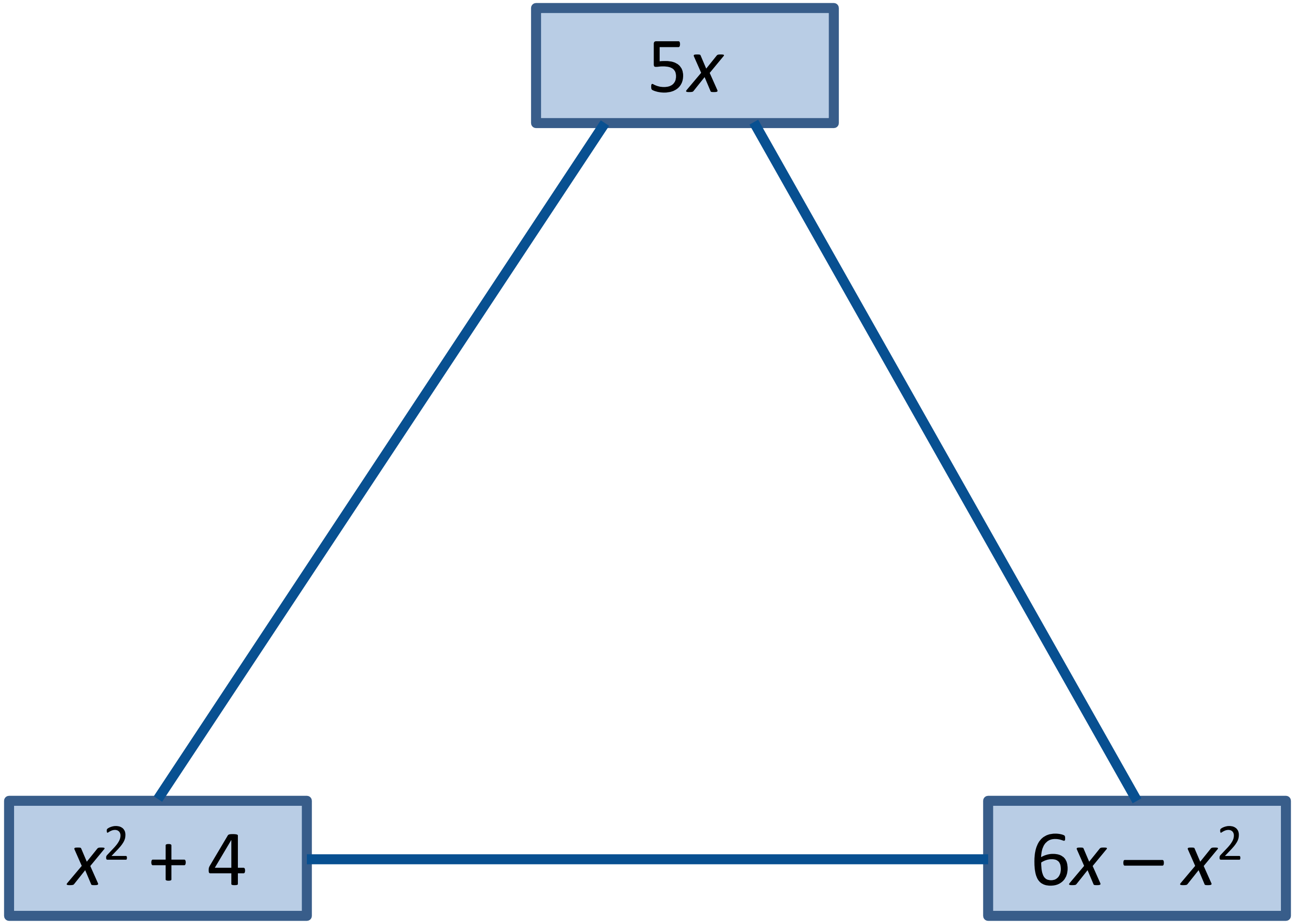
28 $3x - 1 = x + 7$

Have a go!



Foster, C. (2012). Connected expressions. *Mathematics in School, 41*(5), 32–33.


1
2
3


$5x$
$x^2 + 4$
$6x - x^2$

Equations BEFORE Test

Solve these four equations.

Show your method for each one.

$2x + 4 = 3x + 1$	$4x + 7 = 2x - 3$
$5x - 4 = 3x + 6$	$x - 8 = 5x - 20$

Solving Equations

Solve these equations.

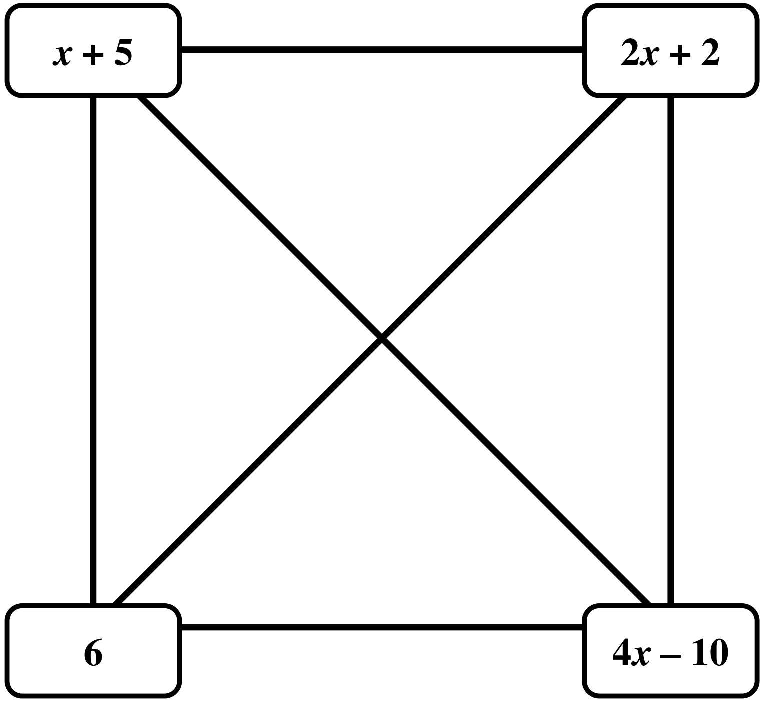
Show your method for each one.

1	$2x + 4 = 3x + 1$	**16**	$7x - 3 = 2x + 2$
2	$3x + 5 = 4x + 3$	**17**	$3x - 5 = x + 1$
3	$4x + 3 = 2x + 5$	**18**	$x + 6 = 2x - 5$
4	$2x - 3 = x - 1$	**19**	$3x - 4 = x - 6$
5	$2x + 1 = 3x - 2$	**20**	$3x + 9 = x - 5$
6	$5x - 3 = 2x + 12$	**21**	$6x - 4 = x + 16$
7	$4x + 9 = 8x - 31$	**22**	$x - 7 = 7x - 25$
8	$2x + 40 = 12x - 110$	**23**	$x + 5 = 4x - 4$
9	$3x + 4 = 5x - 8$	**24**	$6x + 5 = 3x - 7$
10	$2x - 8 = 3x - 16$	**25**	$x + 1 = 7x - 17$
11	$x + 1 = 5x + 9$	**26**	$3x - 4 = 5x + 6$
12	$5x = 2x + 12$	**27**	$8x + 3 = 6x + 15$
13	$9x + 8 = 20 - 3x$	**28**	$x = 20 - x$
14	$5x - 2 = x + 2$	**29**	$3x - 1 = x + 7$
15	$4x + 2 = 3x + 9$	**30**	$x - 6 = 9 - 2x$

Expression Polygons

In the diagram below, every line creates an equation.

So, for example, the line at the top gives the equation $x + 5 = 2x + 2$.



1. Write down and solve the six equations in this diagram.

2. What do you notice about your six solutions?

3. Now make up another diagram like this containing different expressions. Try to make the solutions to your *expression polygon* a "nice" set of numbers.

4. Make up some more *expression polygons* like this and see if other people can solve them.

Instructions to the teacher

“Please allow the two classes the same amount of time to work on these sheets – however much time you have available and feel is appropriate; ideally at least a whole lesson and perhaps more. Help both classes as you would normally, using your professional judgment as to what is appropriate, so that they benefit from the time that they spend on these sheets.”

Equations AFTER Test

Solve these four equations.

Show your method for each one.

$2x + 5 = 3x + 2$	$4x + 5 = 2x - 3$
$5x - 2 = 3x + 8$	$x - 5 = 4x - 20$

Please write down below what you think about the work you have done on solving equations.

Etude 2

Devising Equations

Solving linear equations in which the unknown appears on both sides

Equations Task

1. Make up an equation by choosing numbers to go in the empty boxes.

$$\square x + \square = \square x + \square$$

For example, if you chose the numbers **5**, **4**, **2** and **10**, you would get the equation $5x + 4 = 2x + 10$.

2. Solve your equation.

For example, when you solve the equation $5x + 4 = 2x + 10$ you get $x = 2$.

3. Does your equation have a whole-number answer like this one?

4. Choose another set of four numbers to make another equation.
Try to make as many equations as you can that have whole-number answers.

Etude 3

Enlargements

Enlarging a given shape about a given centre of enlargement with a given scale factor

Enlargement drawings

- “Sir, it’s gone off the edge of the paper!”

- Avoid the problem with pre-prepared sheets

or

- Address the problem by making it the point of the task

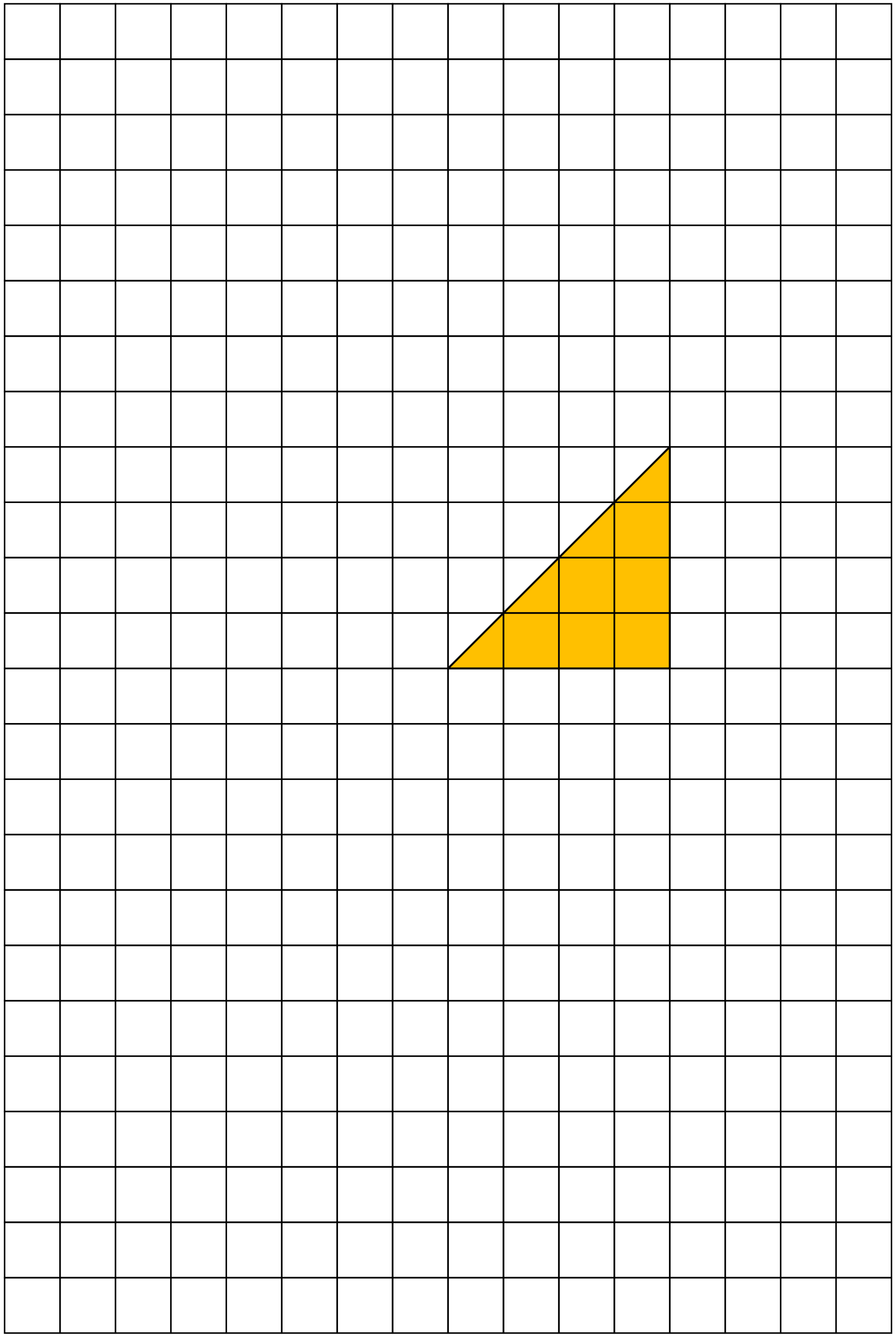
Enlargement drawings

Given an A4 piece of paper and a given shape and a given scale factor of enlargement, where can the centre of enlargement be so that all of the shape stays on the paper?

What is the locus of possible centres of enlargement for the triangle on the sheet if the scale factor is 3?

Foster, C. (2012). Working without a safety net. *The Australian Mathematics Teacher, 68*(2), 25–29.

Foster, C. (2013). Staying on the page. *Teach Secondary*, *3*(1), 57-59.



Have a go!

Enlargement drawings

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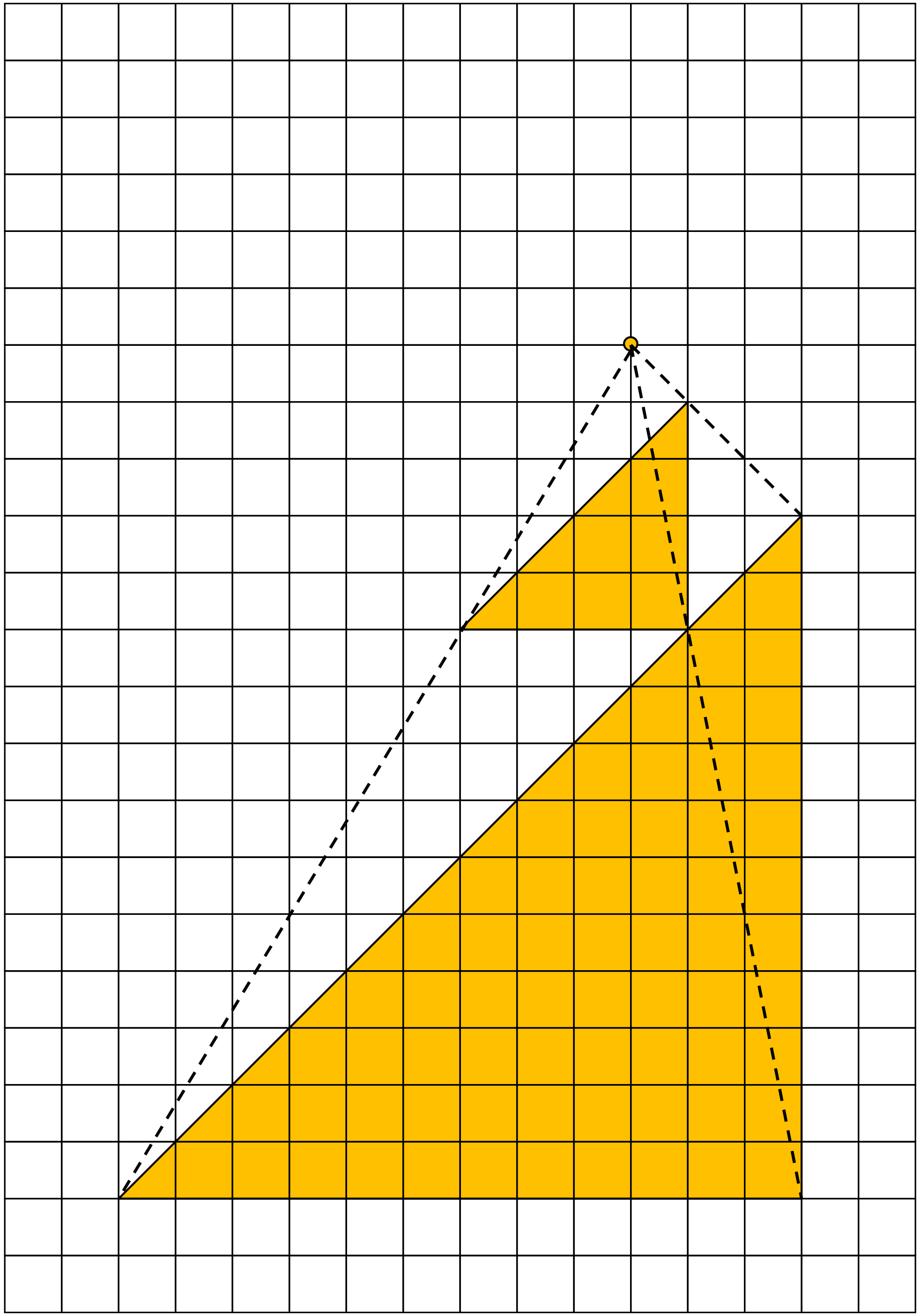
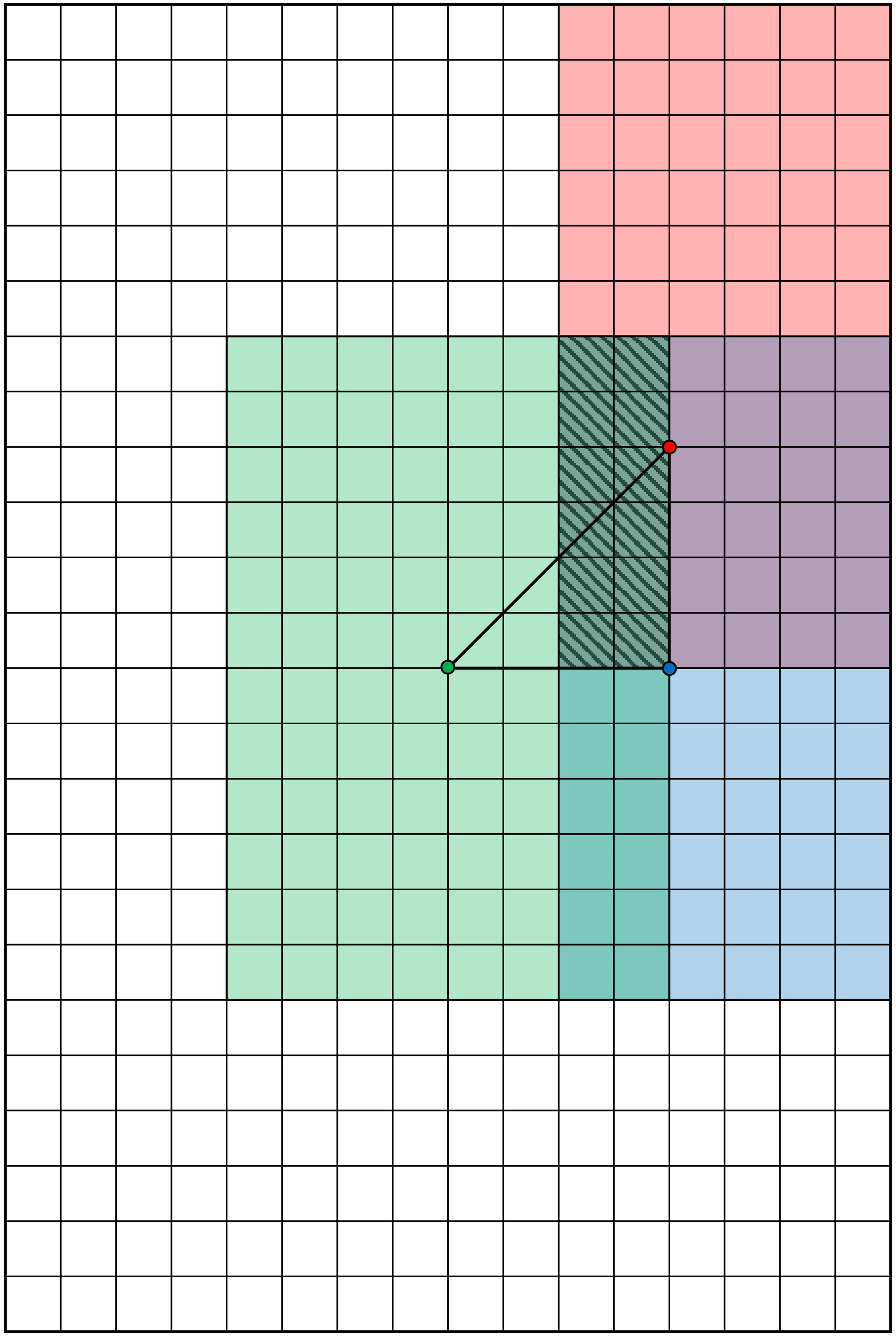
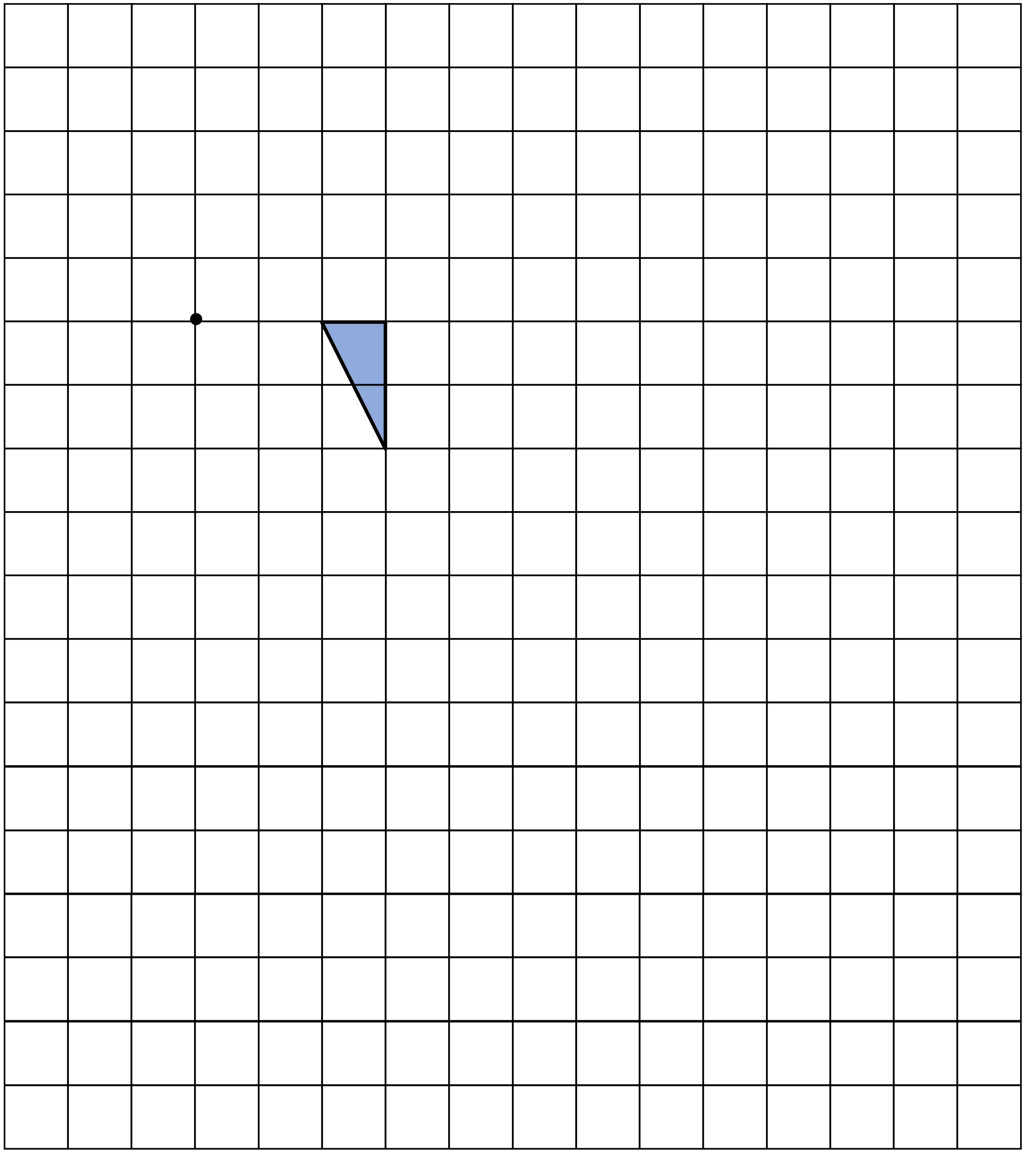




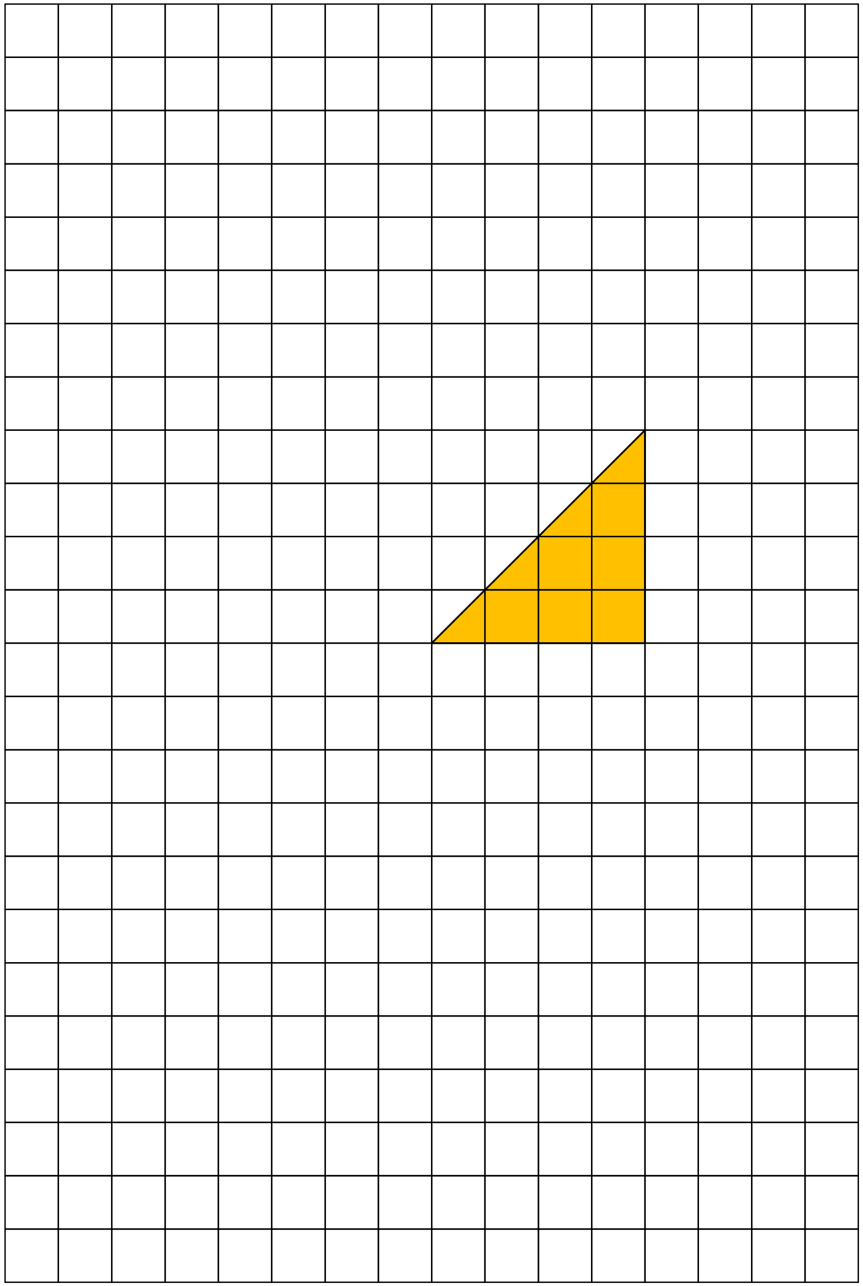
Enlargement BEFORE Test

Enlarge the triangle below with a scale factor of 4 about the centre of enlargement marked with a dot.



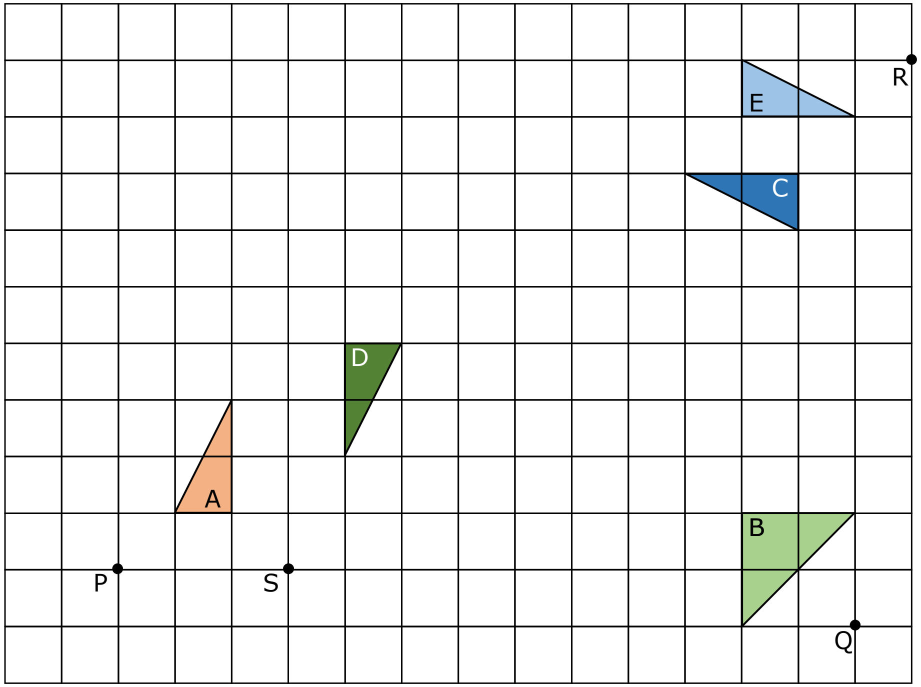
Enlargement Task

For a scale factor 3 enlargement of this triangle, where can the centre of enlargement be so that all of the enlarged shape is on the grid?



Enlargement Exercises

Here are five shapes (A, B, C, D and E) and four points (P, Q, R and S).



1. Enlarge shape A by a scale factor of 3 about centre of enlargement P.
Label your shape F.

2. Enlarge shape B by a scale factor of 2 about centre of enlargement Q.
Label your shape G.

3. Enlarge shape C by a scale factor of 3 about centre of enlargement R.
Label your shape H.

4. Enlarge shape A by a scale factor of 2 about centre of enlargement S.
Label your shape I.

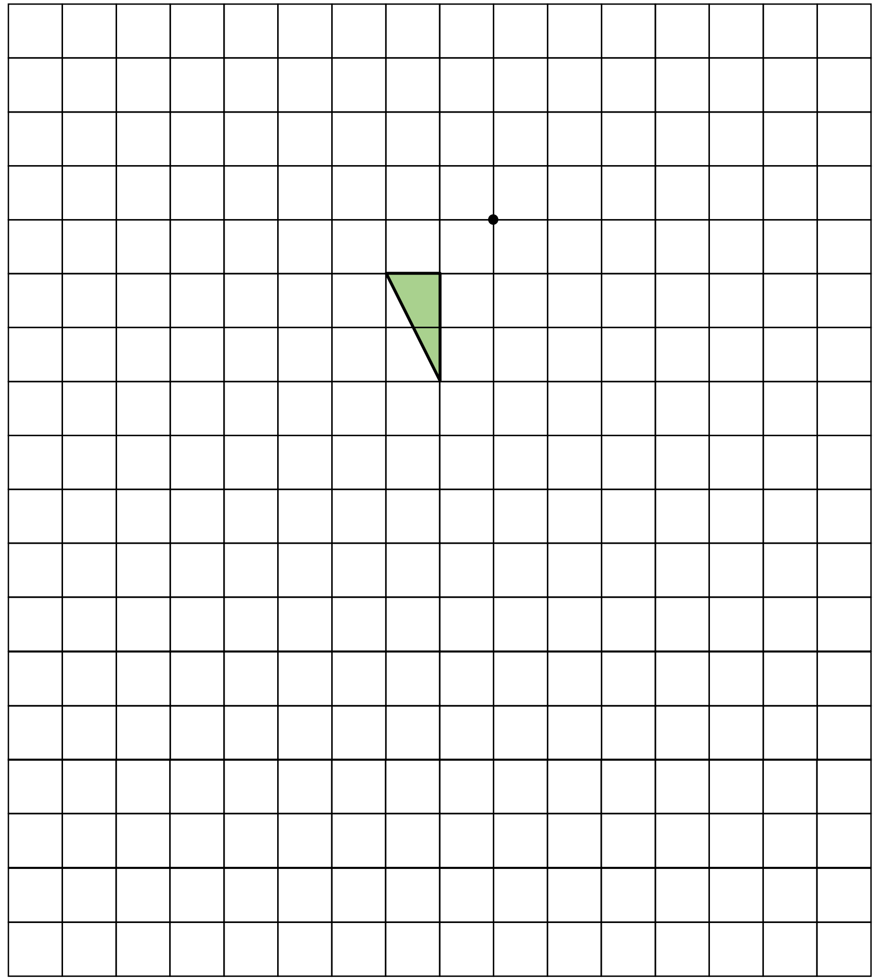
5. Enlarge shape D by a scale factor of 2 about centre of enlargement S.
Label your shape J.

6. Enlarge shape E by a scale factor of 5 about centre of enlargement R.
Label your shape K.

7. Enlarge shape D by a scale factor of 2 about centre of enlargement P.
Label your shape L.

Enlargement AFTER Test

Enlarge the triangle below with a scale factor of 4 about the centre of enlargement marked with a dot.



Please write down below what you think about the work you have done on enlargements.

Summary of Findings

RQ: *Are etudes as effective as traditional exercises at developing students' procedural fluency or not?*

Yes. Evidence that they are equally effective.

- 528 Year 7-9 mathematics students from 11 secondary schools
- Quasi-experimental design, trialling 3 etudes, each against a corresponding traditional exercise
- Statistical analysis gave an overall Bayes factor of 5.83, constituting "substantial" evidence in favour of the null hypothesis of no difference

Conclusion

Even if all you care about is that students develop lots of procedural fluency ...

... you might as well use etudes!

And it is plausible that they have other (harder-to-establish) benefits too:

- Creative investigative inquiry
- Potential for interest and surprise
- Motivation
- Opportunities for communication
- ...

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Acknowledgements

Thank you to the pupils and teachers who participated in the etudes studies.

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